

Symmetry, Constitutive Laws of Bounded Smoothly Deformable Media and Neumann Problems

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Dedicated to Werner Greub on the occasion of his 65th birthday.

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0. Introduction

The purpose of these notes is to show that symmetry, in particular translational symmetry, is the base on which two notions in the theory of elasticity, associated with deformable bodies in \mathbb{R}^n , namely constitutive laws and a certain class of force densities are in a one to one correspondence. This correspondence is given by a von Neumann problem converting force densities into constitutive maps which in turn characterize constitutive laws.

As we will see below any constitutive law F — formulated in our fashion — yields a stress tensor, as in the usual treatment of elasticity (cf. [L, L]). This tensor needs not to be symmetric, however. A criterion for the symmetry is given for constitutive laws invariant and all rotations of \mathbb{R}^n .

To describe in short what we mean by a constitutive law, we begin by looking at a moving deformable bounded body in \mathbb{R}^n . The material should constitute of a deformable medium. The medium forming the boundary may differ from the one forming the inside of the body.

Let us first lay out some elements of the geometric setting. We make the geometric assumption that at any time the body is a n -dimensional, compact, connected, oriented and smooth manifold with (oriented) boundary. The boundary needs not necessarily to be connected. During the motion of the body the diffeomorphism type of the manifold with boundary is assumed to be fixed. These assumptions allow us to think of a standard body M , which from a geometrical point of view is a manifold diffeomorphic to the one moving and deforming in \mathbb{R}^n . Consequently a configuration is a smooth embedding from M into \mathbb{R}^n . The configuration space is hence the collection $E(M, \mathbb{R}^n)$ of all smooth embeddings of M into \mathbb{R}^n . This set equipped with Whitney's C^∞ -topology is a Fréchet manifold (cf. [Bi, Sn, Fi]). A smooth motion of the body in \mathbb{R}^n therefore is described by a smooth curve in $E(M, \mathbb{R}^n)$. The calculus on Fréchet manifolds used in the sequel is the one presented in [Bi, Sn, Fi], which in our setting coincides with the one developed in [Fr, Kr].

The physical qualities of the deforming medium enter certainly the work $F(J)(L)$ needed to deform (infinitesimally) the material at any configuration $J \in E(M, \mathbb{R}^n)$ in any direction L . The directions are tangent vectors to $E(M, \mathbb{R}^n)$. Since the latter space is open in the Fréchet space $C^\infty(M, \mathbb{R}^n)$ of all smooth \mathbb{R}^n -valued functions endowed with the C^∞ -topology (cf. [Hi]), a tangent vector is thus nothing else but a function in $C^\infty(M, \mathbb{R}^n)$ and vice versa.

In the following we take F , which is a one-form on $E(M, \mathbb{R}^n)$, as the basic entity of our notion of a constitutive law. Throughout these notes F is assumed to be smooth. We do not discuss the question as to whether F characterizes the physical properties of the material fully or not.

To allow only internal physical properties of the material to enter F , we have to specify the notion of a constitutive law somewhat further. Basic to this specification is the fact that these sorts of constitutive properties should not be affected by the particular location of the body in \mathbb{R}^n . Thus F has to be invariant under the operation of the translation group \mathbb{R}^n of the real vector space \mathbb{R}^n . Moreover we require $F(J)(L) = 0$, for any constant map L , and any $J \in E(M, \mathbb{R}^n)$ also.

The forms F , which have these two properties, can be regarded as one forms on $\{dJ \mid J \in E(M, \mathbb{R}^n)\}$, where dJ is the differential of J . This set of differentials is equipped with the C^∞ -topology as well and is denoted by $E(M, \mathbb{R}^n)/\mathbb{R}^n$. The latter space is a Fréchet manifold, too. It admits a natural metric of an L_2 -type, which is closely related to the classical Dirichlet integral.

A smooth one-form on $E(M, \mathbb{R}^n)/\mathbb{R}^n$ will be denoted by $F_{\mathbb{R}^n}$. Hence we deal with one-forms of the type $F = d^* F_{\mathbb{R}^n}$ which are supposed to describe the work done under any distortion. To handle such a one-form F we assume that $F_{\mathbb{R}^n}$ can be represented via an integral kernel with respect to the mentioned metric.

A one form F being of the form $F = d^* F_{\mathbb{R}^n}$ and admitting an integral kernel is called a constitutive law.

It turns out that any constitutive law F is determined by some smooth map $\mathfrak{H} \in C^\infty(E(M, \mathbb{R}^n)/\mathbb{R}^n, C^\infty(M, \mathbb{R}^n))$, called a constitutive map.

Hence in our setting we characterize the medium as far as the internal physical properties are encodable in the function \mathfrak{H} . This constitutive function \mathfrak{H} determines at any $dJ \in E(M, \mathbb{R}^n)/\mathbb{R}^n$ two smooth force densities $\Phi(dJ)$ and $\varphi(dJ)$ linked to $\mathfrak{H}(dJ)$ by the following system of equations:

$$\Delta(J)\mathfrak{H}(dJ) = \Phi(dJ)$$

and

$$d\mathfrak{H}(dJ)(N) = \varphi(dJ).$$

Here $\Delta(J)$ is the Laplacian determined by the pull back under J of a fixed scalar product on \mathbb{R}^n . N is the positively oriented unite normal of ∂M in M . The integrability condition necessary to solve this von Neumann problem of which the force densities are given and the function \mathfrak{H} is the unknown is equivalent with the requirement that

$$F(J)(z)=0 \quad \forall J \in E(M, \mathbb{R}^n) \quad \text{and} \quad \forall z \in \mathbb{R}^n.$$

The constitutive function \mathfrak{H} determines a stress tensor T given by

$$T(J)(X, Y) := \langle d\mathfrak{H}(J)X, dJY \rangle \quad \forall J \in E(M, N).$$

X, Y vary among all smooth vector fields on M . Vice versa any stress tensor yields a constitutive map via the force densities mentioned above.

Since F is also affected by the material forming the boundary, we treat in an analogous way the boundary material and exhibit in analogy to \mathfrak{H} a characteristic constitutive map \mathfrak{h} . Thus $\Delta(j)\mathfrak{h}(dJ)$ with $j := J|_{\partial M}$ and $J \in E(M, \mathbb{R}^n)$ describes the force density $\tilde{\varphi}(dJ)$ up to a constant force density along ∂M . However $d\mathfrak{H}(dJ)(N)$ also determines force densities which need not to be of the form $\tilde{\varphi}(dJ)$. This observation allows us to decode the influence of the whole body on the physical quality of the boundary material.

In section 8 we show that both \mathfrak{H} and \mathfrak{h} are structured in the following sense :

In \mathfrak{H} and in \mathfrak{h} is, generically and naturally encoded the work needed to deform volume, area and shape of the body and the boundary respective. The shape is partly expressed by the unite normal vector field $N(j)$ along the embedding j of the boundary. The procedure to decode these influences mentioned is to use an L_2 -splitting of $d\mathfrak{h}(dJ)$.

Finally we discuss the symmetry of the stress tensor, which is based on the action of the rotation group on the configuration space..

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1. Configuration and phase space, geometric preliminaries

Let us think of a material body being deformable moving and deforming in the Euclidean space \mathbb{R}^n . We make the geometric assumption that at any time the body maintains the shape of a n -dimensional, compact, connected, oriented and smooth manifold with (oriented) boundary. The boundary shall not necessarily be connected. The physical qualities of the medium forming the boundary may differ from the ones forming the inside of the body.

Fundamental for our investigation is the assumption that during the motion of the body the diffeomorphism type of the manifold with boundary is assumed to be fixed.

Hence we can think of a standard material body M of which the underlying manifold is diffeomorphic to the body in \mathbb{R}^n .

The standard body constitutes of a deformable medium and we use M to denote both, the manifold with boundary and the material body.

From this situation we read off what we mean by a configuration : A configuration is a smooth embedding

$$J : M \rightarrow \mathbb{R}^n .$$

Hence the space of configurations is $E(M, \mathbb{R}^n)$, the collection of all smooth embeddings of M into \mathbb{R}^n . Endowed with the C^∞ -topology the configuration space is a Fréchet manifold (cf. [Bi, Sn, Fi] or [Hi]).

Clearly each $J \in E(M, \mathbb{R}^n)$ induces a smooth embedding

$$J|_{\partial M} : \partial M \rightarrow \mathbb{R}^n ,$$

a configuration of the boundary ∂M of the body. Let us denote the collection of all smooth embeddings of ∂M into \mathbb{R}^n by $E(\partial M, \mathbb{R}^n)$. The latter space endowed with the C^∞ -topology is also a Fréchet manifold.

Next let us determine the phase space. The set $E(M, \mathbb{R}^n)$ is obviously a subset of $C^\infty(M, \mathbb{R}^n)$, the collection of all smooth \mathbb{R}^n -valued maps of M . We equip it with the C^∞ -topology, also. Since M is compact, $C^\infty(M, \mathbb{R}^n)$ is a complete metrizable locally convex space, a so-called Fréchet space.

The phase space is therefore

$$TE(M, \mathbb{R}^n) = E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) .$$

Proceeding for ∂M as for M we obtain $E(\partial M, \mathbb{R}^n)$ as an open subset of the Fréchet space $C^\infty(\partial M, \mathbb{R}^n)$ (cf.[Hi]). Its tangent bundle is obviously trivial. The phase space for the boundary is hence $E(\partial M, \mathbb{R}^n) \times C^\infty(\partial M, \mathbb{R}^n)$.

In the sequel of these notes we write O_∂ instead of $\{J| \partial M \mid J \in C^\infty(M, \mathbb{R}^n)\}$. The map assigning to any $J \in E(M, \mathbb{R}^n)$ its restriction $J| \partial M$ is called R .

On the configuration space we have two natural actions namely

$$t : E(M, \mathbb{R}^n) \times \mathbb{R}^n \rightarrow E(M, \mathbb{R}^n)$$

and

$$s : SO(n) \times E(M, \mathbb{R}^n) \rightarrow E(M, \mathbb{R}^n)$$

assigning to each $J \in E(M, \mathbb{R}^n)$ and each $z \in \mathbb{R}^n$ the embedding $J + z$ and $g \circ J$ respectively for each $g \in SO(n)$. These actions reflect the translational and the rotational symmetry of $E(M, \mathbb{R}^n)$ respectively.

t and s extend obviously to $C^\infty(M, \mathbb{R}^n)$. The groups \mathbb{R}^n and $SO(n)$ act accordingly on $E(\partial M, \mathbb{R}^n)$. These actions restrict to O_∂ and obviously both extend also to $C^\infty(\partial M, \mathbb{R}^n)$.

The orbit spaces of the respective actions of the translation group \mathbb{R}^n are denoted by $C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n$, $C^\infty(\partial M, \mathbb{R}^n)/\mathbb{R}^n$, $E(M, \mathbb{R}^n)/\mathbb{R}^n$, $E(\partial M, \mathbb{R}^n)/\mathbb{R}^n$ and O_∂/\mathbb{R}^n .

The nature of these spaces are easily understood if we introduce for any $L \in C^\infty(M, \mathbb{R}^n)$ the differential dL which is locally given by the Fréchet derivative. The tangent map TL of L is (L, dL) . The respective notion of $l \in C^\infty(\partial M, \mathbb{R}^n)$ is introduced accordingly.

Hence the orbit spaces mentioned above are nothing else but spaces of differentials of the elements of those spaces on which \mathbb{R}^n acts.

For our later investigations we observe that M and ∂M inherit via respective embeddings into \mathbb{R}^n some basic geometric structures described just below. Let us fix a scalar product \langle, \rangle and a normed determinant function $\underline{\Delta}$ (cf.[Gr]) on \mathbb{R}^n , i.e. \mathbb{R}^n together with \langle, \rangle and $\underline{\Delta}$ is a fixed oriented Euclidean space.

Each $J \in E(M, \mathbb{R}^n)$ and each $j \in E(\partial M, \mathbb{R}^n)$ yield Riemannian metrics $m(J)$ and $m(j)$ on M and ∂M respectively. These metrics are defined by

$$(1.2) \quad m(J)(X, Y) = \langle dJX, dJY \rangle, \quad \forall X, Y \in \Gamma TM$$

and

$$(1.3) \quad m(j)(X, Y) = \langle djX, djY \rangle, \quad \forall X, Y \in \Gamma T\partial M.$$

Here ΓTQ denotes the collection of all smooth vector fields of any smooth manifold Q (with or without boundary). Both $m(J)$ and $m(j)$ depend smoothly on its variables J and j .

Associated with the metrics $m(J)$ and $m(j)$, we have the respective Levi-Civita connections $\nabla(J)$ on M and $\nabla(j)$ on ∂M . They are determined by

$$(1.8) \quad dJ \nabla(J)_X Y = d(dJY)(X), \quad \forall X, Y \in \Gamma TM$$

and

$$(1.9) \quad dj \nabla(j)_X Y = d(djY)(X) - m(j)(W(j)X, Y) \cdot N(j), \quad \forall X, Y \in \Gamma T\partial M.$$

By $W(j)$ we mean the Weingarten map given by

$$(1.10) \quad dN(j)Z = djW(j)Z, \quad \forall Z \in \Gamma TM,$$

where $N(j)$ is the unite normal vector field along j , for which $j^* i_{N(j)}^* \Delta$ determines the orientation class of ∂M .

For any $J \in E(M, \mathbb{R}^n)$ and any $j \in E(M, \mathbb{R}^n)$ let us denote by $\mu(J)$ and $\mu(j)$ the Riemannian volume form determined by $m(J)$ and the orientation of M respectively by $m(j)$ and the orientation of ∂M . Let us denote by N the positively oriented unite normal vector field on ∂M . This vector field depends on $m(J)$.

2. Special one forms on $E(M, \mathbb{R}^n)$ and $E(M, \mathbb{R}^n)/\mathbb{R}^n$

We will characterize the type of the material which constitutes the body M in so far as it affects the work caused by an infinitesimal distortion of M (cf. [He], [E, S], [Bi], [Bi, Sc, So]). This idea is formalized by giving a smooth one-form on $E(M, \mathbb{R}^n)$, i.e. a smooth map

$$(2.1) \quad F : E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \longrightarrow \mathbb{R},$$

which varies linearly in the second argument. We interpret $F(J)(L)$ as the work done if M is distorted by $L \in C^\infty(M, \mathbb{R}^n)$ at the configuration $J \in E(M, \mathbb{R}^n)$. We call the medium described by F a smoothly deformable medium.

Next we expose F to the translational symmetry and require that

$$(2.2) \quad F(J + z) = F(J), \quad \forall J \in E(M, \mathbb{R}^n), \forall z \in \mathbb{R}^n.$$

This means that the work caused by physical processes under consideration does not depend on the particular location of $J(M)$ within \mathbb{R}^n . Moreover a constant distortion by any $z \in \mathbb{R}^n$ is supposed to cause no work. More formally this means that we impose the following restrictions on F basic to our further development :

$$(2.3) \quad (\text{ii.}) \quad F(J)(z) = 0, \quad \forall J \in E(M, \mathbb{R}^n), \forall z \in \mathbb{R}^n.$$

One-forms F on $E(M, \mathbb{R}^n)$ satisfying (2.2) and (2.3) will be the basic ingredients of our notion of constitutive laws. To implement the possibility of extracting force densities from our basic notion of work we need a little more structure associated with our forms satisfying (2.2) and (2.3). We will do this in the next paragraph.

But first let us state the following obvious lemma :

Lemma 2.1 :

A smooth one-form $F : E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfying (2.2) and (2.3) iff it is of the form $F = d^* F_{\mathbb{R}^n}$ that is

$$(2.4) \quad F(J)(L) = F_{\mathbb{R}^n}(dJ)(dL), \quad \forall J \in E(M, \mathbb{R}^n) \text{ and } \forall L \in C^\infty(M, \mathbb{R}^n),$$

where

$$(2.5) \quad F_{\mathbb{R}^n} : E(M, \mathbb{R}^n)/\mathbb{R}^n \times C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n \rightarrow \mathbb{R}$$

is a smooth one-form.

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In the last section we will study the effect of the rotational symmetry on our special types of one-forms, the constitutive laws.

3. The notion of a constitutive law, the Dirichlet integral

The purpose of this section is to define what is meant by an integral representation of a one-form $F_{\mathbb{R}^n}$ on $E(M, \mathbb{R}^n)$ and in turn to define the notion of a constitutive law.

To obtain a tool general enough, we first will introduce a quadric structure on $E(M, \mathbb{R}^n) \times A^1(M, \mathbb{R}^n)$. We denote the collection of all smooth \mathbb{R}^m -valued one-forms of any smooth manifold Q by $A^1(Q, \mathbb{R}^m)$ and equip with the C^∞ -topology in case Q is compact.

Let $\gamma \in A^1(M, \mathbb{R}^n)$ and $J \in E(M, \mathbb{R}^n)$ be given. The two tensor $\langle \gamma, dJ \rangle$ determined by γ and J shall be given by $\langle \gamma X, dJ Y \rangle$ for all $X, Y \in \Gamma TM$. This two tensor $\langle \gamma, dJ \rangle$ yields a unique strong bundle map $A(\gamma, dJ)$ of TM defined by

$$(3.1) \quad \langle \gamma X, dJ Y \rangle = m(J)(A(\gamma, dJ)X, Y), \quad \forall X, Y \in \Gamma TM.$$

From this equation we read off :

$$(3.2) \quad \gamma X = dJ A(\gamma, dJ)X, \quad \forall X \in \Gamma TM.$$

For any pair of one-forms $\gamma_1, \gamma_2 \in A^1(M, \mathbb{R}^n)$ and any embedding $J \in E(M, \mathbb{R}^n)$ we define the dot product of γ_1 and γ_2 relative to J by

$$(3.3) \quad \gamma_1 \cdot \gamma_2 := \text{tr } A(\gamma_1, dJ) \cdot \tilde{A}(\gamma_2, dJ),$$

which is a smooth map on M . Here $\tilde{A}(\gamma_2, dJ)$ is the adjoint of $A(\gamma_2, dJ)$ formed fibre-wise with respect to $m(J)$.

Associated with this product is a quadratic structure $G_{\mathbb{R}^n}(dJ)$ on $A^1(M, \mathbb{R}^n)$ defined by

$$(3.4) \quad G_{\mathbb{R}^n}(dJ)(\gamma_1, \gamma_2) := \int_M \gamma_1 \cdot \gamma_2 \mu(J).$$

As mentioned before $\mu(J)$ denotes the Riemannian volume form determined by $m(J)$ and the given orientation of M . The real number $G_{\mathbb{R}^n}(dJ)(\gamma_1, \gamma_2)$ depends thus smoothly on all its variables dJ, γ_1 and γ_2 .

$G_{\mathbb{R}^n}$ yields obviously a metric on $E(M, \mathbb{R}^n)/\mathbb{R}^n$ again denoted by $G_{\mathbb{R}^n}$.

We say that $F_{\mathbb{R}^n}$, a one-form on $E(M, \mathbb{R}^n)/\mathbb{R}^n$, admits an integral representation if there exists a smooth map

$$\alpha : E(M, \mathbb{R}^n) \rightarrow A^1(M, \mathbb{R}^n),$$

called the kernel of $F_{\mathbb{R}^n}$, such that for any choices of $dJ \in E(M, \mathbb{R}^n)/\mathbb{R}^n$ and $dL \in C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n$

$$(3.5) \quad F_{\mathbb{R}^n}(dJ)(dL) = \int_M \alpha(dJ) \cdot dL \mu(J) = G_{\mathbb{R}^n}(dJ)(\alpha(J), dL)$$

holds true.

We speak of a constitutive law F , if $F = d^* F_{\mathbb{R}^n}$ and $F_{\mathbb{R}^n}$ admits an integral representation with kernel α .

To discuss the uniqueness of the kernel α , associated with a constitutive law, we first prove the following :

Theorem 3.1 :

Let $\gamma \in A^1(M, \mathbb{R}^n)$ and $J \in E(M, \mathbb{R}^n)$. There exists a uniquely determined differential $d\mathfrak{H} \in C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n$ called the exact part of γ and a uniquely determined $\beta \in A^1(M, \mathbb{R}^n)$ such that

$$(3.6) \quad \gamma = d\mathfrak{H} + \beta,$$

where the exact part of β vanishes. Both $d\mathfrak{H}$ and β depend smoothly on J . If $\mathfrak{H}(p_0)$ for some $p_0 \in M$ is kept constant in J , then also \mathfrak{H} varies smoothly in J .

△

Proof :

We will exhibit the proof to some details, in order to reveal the surrounding structure, which will allow us to handle the integral representation introduced in (3.5). First let us construct \mathfrak{H} and β . To this end we fix a basis e_1, \dots, e_n on \mathbb{R}^n , orthonormal with respect to \langle, \rangle . With these data we may write

$$(3.7) \quad \gamma(X) = \sum_{r=1}^n \gamma^r(X) e_r, \quad \forall X \in \Gamma TM,$$

with $\gamma^r \in A^1(M, \mathbb{R}^n)$ for all $r=1, \dots, n$. Since for each r

$$(3.8) \quad \gamma^r(X) = m(J)(Y^r, X), \quad \forall X \in \Gamma TM$$

holds true for a well defined $Y^r \in \Gamma TM$, we find due to Hodge's decomposition (cf. [A, M, R]) a function $\tau^r \in C^\infty(M, \mathbb{R}^n)$ and a uniquely determined vector field $Y_0^r \in \Gamma TM$ such that the following three equations are satisfied

$$(3.9) \quad Y^r = \text{grad}_J \tau^r + Y_0^r$$

and

$$(3.10) \quad \text{div}_J Y_0^r = 0$$

together with the boundary condition
 $m(J)(Y_0^r, N) = 0$ along ∂M .

Here the indices J in grad_J and div_J mean that the respective operations are formed with respect to $m(J)$. This decomposition is obtained by solving the following von Neumann problem

$$(3.11) \quad \Delta(J) \tau^r = -\text{div}_J Y^r$$

with the boundary condition

$$(3.12) \quad d\tau^r(N) = m(J)(Y^r, N).$$

This problem has, according to [Hö], a solution τ^r unique up to a constant.

The desired function \mathfrak{H} and the form β are defined by

$$(3.13) \quad \mathfrak{H} := \sum_{\mathbf{r}} \tau^{\mathbf{r}} \cdot e_{\mathbf{r}}$$

and

$$(3.14) \quad \beta(X) := \sum_{\mathbf{r}} m(J)(Y_0^{\mathbf{r}}, X), \quad \forall X \in \Gamma TM,$$

respectively. It is a matter of routine to show that $d\mathfrak{H}$ and β do not depend on the basis chosen. A somewhat involved matter is to show the smoothness properties. With these notions we immediately deduce (3.6). One easily verifies that the exact part of β vanishes.

□

Some of the calculation made in the proof above allow us to look at $G_{\mathbb{R}^n}(dJ)$ from another angle. Given $\gamma \in A^1(M, \mathbb{R}^n)$ and $J \in E(M, \mathbb{R}^n)$ we have according to (3.7) and (3.8) in above proof

$$(3.15) \quad \gamma(X) = dJA(\gamma, dJ)X = \sum_{r=1}^n m(J)(Y^{\mathbf{r}}, X) e_{\mathbf{r}}, \quad \forall X \in \Gamma TM.$$

Let us denote $(dJ)^{-1}e_{\mathbf{r}}$ by $E_{\mathbf{r}}$, for all $r=1, \dots, n$. Then we read off from the equation (3.15) that

$$(3.16) \quad Y^{\mathbf{r}} = \tilde{A}(\gamma, dJ)E_{\mathbf{r}}, \quad \forall r=1, \dots, n,$$

holds true. This remark yields the following observation :

Proposition 3.2 :

Given $\gamma_1, \gamma_2 \in A^1(M, \mathbb{R}^n)$, $J \in E(M, \mathbb{R}^n)$ and a fixed basis e_1, \dots, e_n on \mathbb{R}^n orthonormal with respect to \langle, \rangle , then there exist two sets

$$Y_1^1, \dots, Y_1^n \quad \text{and} \quad Y_2^1, \dots, Y_2^n$$

of vector fields in ΓTM , such that

$$(3.17) \quad \gamma_1 \cdot \gamma_2 = \sum_{r=1}^n m(J)(Y_1^r, Y_2^r)$$

and hence

$$(3.18) \quad G_{\mathbb{R}^n}(dJ)(\gamma_1, \gamma_2) := \int_M \gamma_1 \cdot \gamma_2 \mu(J) = \sum_{r=1}^n \int_M m(J)(Y_1^r, Y_2^r) \mu(J).$$

If in addition $\gamma_1 = d\mathfrak{H}$ for some $\mathfrak{H} \in C^\infty(M, \mathbb{R}^n)$ then $G_{\mathbb{R}^n}(dJ)(d\mathfrak{H}, \gamma_2) = 0$ provided that the exact part of γ_2 vanishes.

△

Proof :

Let $Y_i^r \in \Gamma TM$ for $r=1, \dots, n$ and $i=1, 2$ be as in (3.8). Then

$$(3.19) \quad \begin{aligned} \gamma_1 \cdot \gamma_2 &= \text{tr } A(\gamma_1, dJ) \cdot \tilde{A}(\gamma_2, dJ) = \sum_{r=1}^n m(J)(A(\gamma_1, dJ) \cdot \tilde{A}(\gamma_2, dJ) E_r, E_r) \\ &= \sum_{r=1}^n m(J)(Y_1^r, Y_2^r) \end{aligned}$$

establishing (3.17). To show the last part of the proposition we use Gauss' theorem under the three assumptions that $\mathfrak{H} = \sum_{r=1}^n \tau^r e_r$ and $\text{div}_J Y_0^r = 0$ as well as $m(J)(Y_0^r, N) = 0$:

$$(3.20) \quad \begin{aligned} G_{\mathbb{R}^n}(dJ)(\gamma_1, \gamma_2) &= \sum_{r=1}^n \int_M m(J)(\text{grad}_J \tau^r, Y_0^r) \\ &= \sum_{r=1}^n \int_M d\tau^r(Y_0^r) \mu(J) \\ &= \sum_{r=1}^n \int_M (\text{div}_J(\tau^r Y_0^r) - \tau \cdot \text{div}_J Y_0^r) \mu(J) \\ &= \sum_{r=1}^n \int_M m(J)(\tau^r Y_0^r, N) \mu(J) \\ &= 0. \end{aligned}$$

□

In case γ_1 and γ_2 in the above proposition are exact, then the respective vector fields in (3.20) are gradients. Hence the right hand side of the integral in (3.18) is the classical Dirichlet integral (cf.[J]) for \mathbb{R}^n -valued functions. We call the middle integral in (3.18), therefore the Dirichlet integral of any two smooth \mathbb{R}^n -valued forms γ_1, γ_2 relative to $J \in E(M, \mathbb{R}^n)$.

Proposition 3.2 also shows that the integral kernel of a constitutive law is not unique at all. We may add to any kernel a map which assumes as its values, one-forms of which the exact part vanishes. However the following theorem guaranties us a uniqueness of a very specific type of kernel. The proof of the following theorem is a matter of routine :

Theorem 3.3 :

Let F be a constitutive law with integral kernel. There exists a unique smooth map

$$(3.21) \quad \alpha : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n) / \mathbb{R}^n \subset A^1(M, \mathbb{R}^n) ,$$

such that for any $J \in E(M, \mathbb{R}^n)$ and any $L \in C^\infty(M, \mathbb{R}^n)$

$$(3.22) \quad F(J)(L) = \int_M \alpha(dJ) \cdot dL \, \mu(J) = G_{\mathbb{R}^n}(dJ)(\alpha(dJ), dL)$$

holds true. In fact there is a unique smooth map

$$\mathfrak{H} : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n) ,$$

satisfying the following two equations

$$(3.23) \quad \alpha(dJ) = d\mathfrak{H}(dJ) , \quad \forall dJ \in E(M, \mathbb{R}^n) / \mathbb{R}^n$$

and

$$(3.24) \quad \int_{\partial M} \langle \mathfrak{H}(dJ), z \rangle \mu(J) = 0 , \quad \forall z \in \mathbb{R}^n .$$

Equation (3.22) together with (3.23) show that $d\mathfrak{H}$ is the generalized gradient of $F_{\mathbb{R}^n}$ with respect to $G_{\mathbb{R}^n}$.

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It is this type of gradient mentioned above which characterizes the material.

Clearly each constitutive law F forms via α and equation (3.1) a tensor on M given by

$$(3.25) \quad \begin{aligned} T(J)(X, Y) &:= \langle \alpha(dJ)X, dJY \rangle \\ &= m(J)(dJA(\alpha(dJ), dJX), Y), \quad \forall X, Y \in \Gamma TM. \end{aligned}$$

Vice versa each two-tensor depending smoothly on dJ defines a kernel of a constitutive law. We call $T(J)$ the stress tensor at the configuration J .

$T(J)$ splits into a symmetric and a skew symmetric part. We will investigate at the end of these notes under what conditions on $d\mathfrak{H}$ the skew symmetric part vanishes.

4. Force densities associated with constitutive laws

The purpose of this section is to associate with any constitutive law at any configuration some well defined force densities, one acting upon the whole body, and an extra one acting upon the boundary only. Vice versa any given pair of such force densities satisfying an integrability condition will be obtained via a suitable constitutive law.

Throughout this section F is a constitutive law, its kernel is called α . By the previous theorem we may assume that $\alpha(E(M, \mathbb{R}^n)/\mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)/\mathbb{R}^n$.

To construct the force densities mentioned we use F in the form

$$(4.1) \quad F(J)(L) = \int_M \text{tr } A(\alpha(dJ), dJ) \cdot \tilde{A}(dL, dJ) \mu(J),$$

holding for any of the variables of F . Representing any $L \in C^\infty(M, \mathbb{R}^n)$ relative to a given $J \in E(M, \mathbb{R}^n)$ in the form

$$(4.2) \quad L = dJ X(L, J),$$

with a unique $X(L, J) \in \Gamma TM$ we derive

$$(4.3) \quad dL X = dJ \nabla(J)_X X(L, J), \quad \forall X \in \Gamma TM.$$

and hence obtain immediately

$$(4.4) \quad A(dL, dJ) = \nabla(J) X(L, J), \quad \forall L \in C^\infty(M, \mathbb{R}^n).$$

Thus if e_1, \dots, e_n is a orthonormal basis of \mathbb{R}^n for which we let $E_r \in \Gamma TM$ again be given by $dJ E_r = e_r$ for $r=1, \dots, n$ then

$$(4.5) \quad F(J)(L) = \sum_{r=1}^n \int_M m(J) (\tilde{A}(\alpha(dJ), dJ) \cdot \nabla(J)_{E_r} X(L, J), E_r) \mu(J).$$

Let us introduce the notion $\operatorname{div}_J T$, the divergence of a strong bundle endomorphism T of TM by

$$(4.6) \quad \operatorname{div}_J T := \sum_{r=1}^n \nabla(J)_{E_r} (T)(E_r).$$

This notion does not depend of the basis chosen. Equation (4.6) together with (4.5) imply

$$(4.7) \quad F(J)(L) = \int_M \operatorname{div}_J (\tilde{A}(\alpha(dJ), dJ) X(L, J)) \mu(J) \\ - \int_M m(J) (\operatorname{div}_J A(\alpha(dJ), dJ), X(L, J)) \mu(J).$$

To bring these formulas in a more familiar form we introduce the notions of $\Delta(J)K$ and $\Delta(J)\gamma$, the Laplacian for any $K \in C^\infty(M, \mathbb{R}^n)$ and any $\gamma \in A^1(M, \mathbb{R}^n)$. In doing so we follow [Mat]. We begin with the definition of δ by letting

$$(4.8) \quad \delta K = 0.$$

If $\gamma \in A^1(M, \mathbb{R}^m)$ for some natural number m , we set

$$(4.9) \quad \delta\gamma = -\sum_{r=1}^n \nabla(J)_{E_r}(\gamma)(E_r) .$$

Clearly

$$(4.10) \quad \delta\gamma = -\operatorname{div}_J Y ,$$

if

$$\gamma(X) = m(J)(Y, X) , \quad \forall X \in \Gamma TM.$$

$\Delta(J)$ is then defined by

$$(4.11) \quad \Delta(J) := d\delta + \delta d .$$

Consequently we have

$$(4.12) \quad \Delta(J)K = \delta dK = -\sum_{r=1}^n \nabla(J)_{E_r}(dK)(E_r) .$$

and therefore we verify the following relation :

$$(4.13) \quad \begin{aligned} \Delta(J)K &= -\left(\sum_{r=1}^n d(dJ A(dK, dJ E_r))(E_r) - dJ A(dK, dJ)(\nabla(J)_{E_r} E_r)\right) \\ &= -\sum_{r=1}^n dJ \nabla(J)_{E_r}(A(dK, dJ))E_r \\ &= -dJ \operatorname{div}_J A(dK, dJ) . \end{aligned}$$

Hence equation (4.7) turns into

$$(4.14) \quad \begin{aligned} F(J)(L) &= \int_M \operatorname{div}_J(\tilde{A}(\alpha(dJ), dJ)X(L, J)) \mu(J) \\ &\quad + \int_M \langle \Delta(J)\mathfrak{H}(dJ), L \rangle \mu(J) , \end{aligned}$$

with $\alpha(dJ) = d\mathfrak{H}(dJ)$ for some $\mathfrak{H} \in (C^\infty(E(M, \mathbb{R}^n))/\mathbb{R}^n, C^\infty(M, \mathbb{R}^n))$.

Using Gauss' theorem we derive with the help of theorem 3.3 the following

Proposition 4.1 :

Let F be a constitutive law. Then for each $J \in E(M, \mathbb{R}^n)$ there exists a smooth map

$$\mathfrak{H} : E(M, \mathbb{R}^n) \longrightarrow C^\infty(M, \mathbb{R}^n)$$

uniquely determined up to a smooth map from $E(M, \mathbb{R}^n)$ into \mathbb{R}^n for which

$$(4.15) \quad \begin{aligned} F(J)(L) = & \int_M \langle \Delta(J) \mathfrak{H}(dJ), L \rangle \mu(J) \\ & + \int_{\partial M} \langle d\mathfrak{H}(dJ)(N), L \rangle i_N \mu(J) \end{aligned}$$

and in turn a Green's equation

$$(4.16) \quad \begin{aligned} & \int_M \langle \Delta(J) \mathfrak{H}(dJ), L \rangle \mu(J) - \int_M \langle \mathfrak{H}(dJ), \Delta(J)L \rangle \mu(J) \\ & = \int_{\partial M} \langle dL(N), \mathfrak{H}(dJ) \rangle i_N \mu(J) - \int_{\partial M} \langle d\mathfrak{H}(dJ)(N), L \rangle i_N \mu(J) \end{aligned}$$

are valued for all variables of F . Here $i_N \mu(J)$ is the volume element on ∂M defined by $\mu(J)$ and N , the positively oriented unite normal vector field along $\partial M \subset M$.

Δ

The developments made so far show that any characteristic of the material formulated via our notion of a constitutive law are encoded in the map \mathfrak{H} . We call this map \mathfrak{H} therefore a constitutive map.

The above proposition motivates us to set for any $J \in E(M, \mathbb{R}^n)$

$$(4.17) \quad \Phi(J) := \Delta(J) \mathfrak{H}(dJ)$$

and

$$(4.18) \quad \varphi(dJ) := d\mathfrak{H}(dJ)(N),$$

with $\mathfrak{H}(dJ)$ as in (4.15).

We call the maps Φ and φ the force densities associated with F .

To prove the following theorem is now a matter of routine again.

Theorem 4.2 :

Every constitutive law $F \in A^1(E(M, \mathbb{R}^n), \mathbb{R})$ admits a smooth constitutive map

$$(4.19) \quad \mathfrak{H} : E(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n),$$

uniquely determined up to maps in $C^\infty(E(M, \mathbb{R}^n)/\mathbb{R}^n, \mathbb{R}^n)$, such that F can be expressed by

$$(4.20) \quad F(J)(L) = \int_M \langle \Delta(J) \mathfrak{H}(dJ), L \rangle \mu(J) + \int_{\partial M} \langle d\mathfrak{H}(dJ)(N), L \rangle i_N \mu(J),$$

for each $J \in E(M, \mathbb{R}^n)$ and each $L \in C^\infty(M, \mathbb{R}^n)$. For all $J \in E(M, \mathbb{R}^n)$ the map \mathfrak{H} defines the force densities

$$(4.21) \quad \Phi(dJ) = \Delta(J) \mathfrak{H}(dJ)$$

and

$$(4.22) \quad \varphi(dJ) = d\mathfrak{H}(dJ)(N),$$

with the following property :

$$(4.23) \quad 0 = \int_M \langle \Phi(dJ), z \rangle \mu(J) + \int_{\partial M} \langle \varphi(dJ), z \rangle i_N \mu(J), \quad \forall z \in \mathbb{R}^n.$$

Given vice versa two smooth maps

$$(4.24) \quad \Phi : E(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n)$$

and

$$(4.25) \quad \varphi : E(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow C^\infty(\partial M, \mathbb{R}^n)$$

for which the integrability condition (4.23) holds. Then there exists a smooth map

$$\mathfrak{H} : E(M, \mathbb{R}^n)/\mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n)$$

such that $\mathfrak{H}(dJ)$ is uniquely determined up to a constant for each $J \in E(M, \mathbb{R}^n)$ and which satisfies (4.21) and (4.22). Moreover \mathfrak{H} is a constitutive map for the constitutive law F given by

$$(4.26) \quad F(J)(L) = \int_M \langle \Phi(dJ), L \rangle \mu(J) + \int_{\partial M} \langle \varphi(dJ), L \rangle i_N \mu(J),$$

for all $J \in E(M, \mathbb{R}^n)$ and for all $L \in C^\infty(M, \mathbb{R}^n)$.

△

Remark 4.3

This theorem shows us that our notion of constitutive laws based on translational invariance are equivalent with the \mathbb{R}^n -valued solution of von Neumann problems formulated on M and hence are equivalent with a pair of force densities satisfying the integrability condition (4.23), which is obviously the analogon of (2.3).

Moreover if for each $J \in E(M, \mathbb{R}^n)$ the one form $\alpha(dJ)$ splits as

$$\alpha(dJ) = d\mathfrak{H}(dJ) + \beta(dJ)$$

with $d\mathfrak{H}(dJ)$ the exact part of $\alpha(dJ)$, then both $\alpha(dJ)$ and $d\mathfrak{H}(dJ)$ determine the same force densities !

5. Constitutive laws for the boundary

The task in this section is to study constitutive laws for the boundary, that is for a deformable medium forming a skin of which the underlying point set is the $(n-1)$ -dimensional manifold ∂M . This skin, formed by a deformable material, will be studied on its own and is not regarded as boundary of some body. In doing so, we first formulate in analogy to sections two and three what is meant by a constitutive law for a skin. Also in this case the translational invariance and an appropriate integral representation of one-forms describing the work of the material subjected to distortions is the essential tool.

Let us recall that the open set $O_\partial \subset E(\partial M, \mathbb{R}^n)$ is the collection of all $J|_{\partial M}$ with $J \in E(M, \mathbb{R}^n)$. The constitutive laws to introduce will be formulated on any open set $O \subset E(M, \mathbb{R}^n)$ and will later be specified on O_∂ .

At the very first we introduce the notion corresponding to the Dirichlet integral :
Given any $l \in C^\infty(\partial M, \mathbb{R}^n)$ and any $j \in E(\partial M, \mathbb{R}^n)$ then for all $X, Y \in \Gamma T \partial M$

$$(5.1) \quad \langle dl X, dj Y \rangle = m(j)(A(dl, dj)X, Y)$$

holds for some smooth strong bundle endomorphism $A(dl, dj)$ of $T\partial M$. Moreover there is a uniquely defined smooth map

$$(5.2) \quad c(dl, dj) : \partial M \rightarrow so(n),$$

with $so(n)$ the linear space of all skew maps of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, satisfying the following two conditions

$$(5.3) \quad c(dl, dj)dj(T_p \partial M) \subset \mathbb{R} \cdot N(j)(p), \quad \forall p \in \partial M$$

and

$$(5.4) \quad c(dl, dj)N(j)(p) \subset dj T_p \partial M, \quad \forall p \in \partial M,$$

and such that the equation

$$(5.5) \quad dl X = c(dj, dl)dj X + dj A(dl, dj)X$$

holds true for any $X \in \Gamma TM$. We refer to [Bi, Sn, Fi] or [Bi, Sc, So] for more details. Based on (5.4) we introduce $U(dl, dj)$ by

$$(5.6) \quad c(dl, dj)N(j) = dj U(dl, dj).$$

This vector field $U(dl, dj) \in \Gamma T\partial M$ is obviously uniquely determined.

Splitting $A(dl, dj)$ into its skew— respectively selfadjoint parts $C(dl, dj)$ and $B(dl, dj)$ formed pointwise with respect to $m(j)$ we end up with

$$(5.7) \quad dl = c(dl, dj) \cdot dj + dj(C(dl, dj) + B(dl, dj)).$$

This decomposition relative to any $dj \in E(\partial M, \mathbb{R}^n)$ generalizes in the obvious way to any $\gamma \in A^1(\partial M, \mathbb{R}^n)$ and reads in this case as

$$(5.7a) \quad \gamma = c(\gamma, dj) \cdot dj + dj(C(\gamma, dj) + B(\gamma, dj)).$$

The quadratic structure $G_{\mathbb{R}^n}^{\partial}(dj)$ at $dj \in E(\partial M, \mathbb{R}^n)/\mathbb{R}^n$ applied to any two $\gamma_1, \gamma_2 \in A^1(\partial M, \mathbb{R}^n)$ is defined by integrating the function

$$\begin{aligned}
 (5.8) \quad \gamma_1 \cdot \gamma_2 &:= -\frac{1}{2} \operatorname{tr} c(\gamma_1, dj) \cdot c(\gamma_2, dj) \\
 &\quad - \operatorname{tr} C(\gamma_1, dj) \cdot C(\gamma_2, dj) \\
 &\quad + \operatorname{tr} B(\gamma_1, dj) \cdot B(\gamma_2, dj)
 \end{aligned}$$

with respect to $\mu(j)$, that is, it is defined by

$$\begin{aligned}
 (5.9) \quad G_{\mathbb{R}^n}^{\partial}(dj)(\gamma_1, \gamma_2) &:= \int_M \gamma_1 \cdot \gamma_2 \mu(j) \\
 &= -\frac{1}{2} \int_M \operatorname{tr} c(\gamma_1, dj) \cdot c(\gamma_2, dj) \mu(j) \\
 &\quad - \int_M \operatorname{tr} C(\gamma_1, dj) \cdot C(\gamma_2, dj) \mu(j) \\
 &\quad + \int_M \operatorname{tr} B(\gamma_1, dj) \cdot B(\gamma_2, dj) \mu(j).
 \end{aligned}$$

$G_{\mathbb{R}^n}^{\partial}$ obviously yields a metric on $E(M, \mathbb{R}^n)$. Let $O \subset E(\partial M, \mathbb{R}^n)$ be any open set.

We now define a constitutive law F_{∂} on O in analogy to section two, that is we require

$$(5.10) \quad F_{\partial} = d^* F_{\mathbb{R}^n}^{\partial},$$

to hold for some one-form $F_{\mathbb{R}^n}^{\partial}$ on O/\mathbb{R}^n and demand furthermore that for some $\alpha \in C^{\infty}(O, A^1(\partial M, \mathbb{R}^n))$ the following equation is valid

$$(5.11) \quad F_{\partial}(dj)(dl) = \int_M \alpha(dj) \cdot dl \mu(j), \quad \forall dl \in C^{\infty}(\partial M, \mathbb{R}^n)|_{\mathbb{R}^n}, \forall dj \in O/\mathbb{R}^n.$$

We introduce for any $j \in E(\partial M, \mathbb{R}^n)$ the Laplacian $\Delta(j)$ accordingly to (4.11) but require that E_s in this case is a moving frame on ∂M .

With this notion at hand the constitutive laws on O are characterized in details in the next theorem :

Theorem 5.1 :

Let F_{∂} be a constitutive law on any open set $O \subset E(\partial M, \mathbb{R}^n)$. The following are then equivalent :

(i.) F_{∂} admits a kernel $\alpha \in C^{\infty}(O/\mathbb{R}^n, A^1(\partial M, \mathbb{R}^n))$.

(ii.) There is a smooth map $h \in C^{\infty}(O/\mathbb{R}^n, C^{\infty}(\partial M, \mathbb{R}^n))$ uniquely determined up to maps in $C^{\infty}(O/\mathbb{R}^n, \mathbb{R}^n)$, such that

$$(5.12) \quad F_{\partial}(j)(l) = \int_{\partial M} dh(dj) \cdot dl \, \mu(j), \quad \forall j \in O, \\ \text{and } \forall l \in C^{\infty}(\partial M, \mathbb{R}^n).$$

(iii.) There is a smooth map $h \in C^{\infty}(O/\mathbb{R}^n, C^{\infty}(\partial M, \mathbb{R}^n))$ uniquely determined up to maps in $C^{\infty}(E(M, \mathbb{R}^n), \mathbb{R}^n)$, such that

$$(5.13) \quad F_{\partial}(j)(l) = \int_{\partial M} \langle \Delta(j)h(dj), l \rangle \mu(j), \quad \forall j \in O, \forall l \in C^{\infty}(\partial M, \mathbb{R}^n).$$

(iv.) There is a unique smooth map $\varphi \in C^{\infty}(O, C^{\infty}(\partial M, \mathbb{R}^n))$, such that

$$(5.14) \quad F_{\partial}(j)(l) = \int_{\partial M} \langle \varphi(dj), l \rangle \mu(j), \quad \forall j \in O, \forall l \in C^{\infty}(\partial M, \mathbb{R}^n),$$

and which satisfies

$$(5.15) \quad \int_{\partial M} \langle \varphi(dj), z \rangle \mu(j) = 0, \quad \forall j \in O, \forall z \in \mathbb{R}^n.$$

Δ

For an explicit formulation of $\Delta(j)$ in terms of the coefficients of dj we refer to [Bi, Sc, So].

Remark :

The remark at the end of section 4 translates accordingly to the situation studied in this section.

Next we will investigate a boundary material implemented to a body formed by a deformable medium.

6. The interplay between constitutive laws of boundary and body

The deformable media forming the inside of the body and the boundary respectively may differ and each separate material hence has to be described on one hand by different constitutive laws. This we have done in the previous sections. On the other hand these materials together form one body and should be describable by only one constitutive law holding for the whole body. Since constitutive laws behave additively, the comparison between the two procedures allows us to decode the influence of the whole body to the constitutive properties of the boundary material.

Let the constitutive law of the deformable medium forming the whole body be called by F again.

According to theorem 4.2, F is determined by a smooth constitutive map

$$\mathfrak{H} : E(M, \mathbb{R}^n) / \mathbb{R}^n \rightarrow C^\infty(M, \mathbb{R}^n) .$$

The following theorem exhibits its influence to the constitutive entities of the material forming the boundary of the body. We will indicate next the methods to prove it.

The map \mathfrak{H} yields according to theorem 4.2 force densities

$$(6.1) \quad \Phi : E(M, \mathbb{R}^n) / \mathbb{R}^n \rightarrow C^\infty(M, \mathbb{R}^n)$$

and

$$(6.2) \quad \varphi : E(M, \mathbb{R}^n) / \mathbb{R}^n \rightarrow C^\infty(\partial M, \mathbb{R}^n)$$

The force density acting on ∂M , is defined by

$$(6.3) \quad \varphi(dJ) = d\mathfrak{H}(dJ)(N) , \quad \forall dJ \in E(M, \mathbb{R}^n) / \mathbb{R}^n .$$

Having the integrability condition (5.14) of $\Delta(J)$ in mind, we split this force density φ into

$$(6.4) \quad \varphi(dJ) = \varphi_{\mathbb{R}^n}(dJ) + \psi(dJ) , \quad \forall dJ \in E(M, \mathbb{R}^n) / \mathbb{R}^n ,$$

where $\varphi_{\mathbb{R}^n}(dJ)$ is characterized for each $dJ \in E(M, \mathbb{R}^n) / \mathbb{R}^n$ by the equation

$$(6.5) \quad \int_{\partial M} \langle \varphi_{\mathbb{R}^n}(dJ), z \rangle i_N \mu(J) = 0, \quad \forall z \in \mathbb{R}^n$$

and where

$$\psi : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

is a smooth map, which makes (6.4) to hold.

According to theorem 5.1 the condition (6.5) allows us to choose some map

$$(6.6) \quad \mathfrak{h} : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow C^\infty(\partial M, \mathbb{R}^n),$$

such that for all $dJ \in E(M, \mathbb{R}^n) / \mathbb{R}^n$ the equation

$$(6.7) \quad \varphi_{\mathbb{R}^n}(dJ) = \Delta(J| \partial M) \mathfrak{h}(dJ)$$

holds true.

Theorem 6.1 :

Any smoothly deformable medium is characterized by a constitutive map

$$\mathfrak{H} : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow C^\infty(M, \mathbb{R}^n),$$

determining itself two smooth maps

$$(6.8) \quad \mathfrak{h} : E(M, \mathbb{R}^n) / \mathbb{R}^n \longrightarrow C^\infty(\partial M, \mathbb{R}^n)$$

and

$$(6.9) \quad \psi : E(M, \mathbb{R}^n) \longrightarrow \mathbb{R}^n,$$

which are linked to \mathfrak{H} by the boundary condition

$$(6.10) \quad d\mathfrak{H}(dJ)(N) = \Delta(J| \partial M) \mathfrak{h}(dJ) + \psi(dJ).$$

\mathfrak{h} is unique up to \mathbb{R}^n -valued smooth maps of $E(M, \mathbb{R}^n)$ and ψ is unique. Moreover \mathfrak{H} satisfies the integrability conditions

$$(6.11) \quad 0 = \int_M \langle \Delta(J) \mathfrak{H}(dJ), z \rangle \mu(J) + \int_{\partial M} \langle d\mathfrak{H}(dJ)(N), z \rangle i_N \mu(J),$$

for each $J \in E(M, \mathbb{R}^n)$ and each $z \in \mathbb{R}^n$. Equation (6.11) equivalently formulated reads as

$$(6.12) \quad 0 = \int_M \langle \Delta(J) \mathfrak{H}(dJ), z \rangle \mu(J) + \int_{\partial M} \langle \psi(dJ), z \rangle i_N \mu(J),$$

a boundary condition holding for \mathfrak{H} and ψ together. The constitutive law on $E(M, \mathbb{R}^n)$ describing the constitutive properties of the materials forming body together with its boundary is thus given via the formula

$$(6.13) \quad F(J)(L) = \int_M \langle \Delta(J) \mathfrak{H}(J), L \rangle \mu(J) + \int_{\partial M} \langle \Delta(J|_{\partial M}) \mathfrak{h}(J|_{\partial M}) + \psi(dJ), L \rangle i_N \mu(J),$$

$$\forall J \in E(M, \mathbb{R}^n), \forall L \in C^\infty(M, \mathbb{R}^n).$$

The work of any distortion $l \in C^\infty(\partial M, \mathbb{R}^n)$ of the deformable material forming the boundary detached from the body is for any $J \in E(M, \mathbb{R}^n)$ given by

$$(6.14) \quad F_{\partial M}(dJ)(l) = \int_{\partial M} \langle \Delta(J|_{\partial M}) \mathfrak{h}_{\partial}(dJ), l \rangle i_N \mu(J).$$

for some constitutive map $\mathfrak{h}_{\partial} \in C^\infty(O_{\partial}/\mathbb{R}^n, C^\infty(\partial M, \mathbb{R}^n))$.

The constitutive properties of the deformable medium of the boundary detached from the body, which is given by a smooth map $\mathfrak{h}_{\partial} \in C^\infty(O_{\partial}/\mathbb{R}^n, C^\infty(\partial M, \mathbb{R}^n))$. Hence $\mathfrak{h} - \mathfrak{h}_{\partial}$ and ψ describe how the constitutive properties of the material forming the boundary of the body is affected by the fact that this material is incorporated into the material forming the whole body.

△

7. Simple examples

In this section we will study well known one forms on $E(M, \mathbb{R}^n)$ in the light of our formalism developed above. We do so by looking at particularly simple constitutive maps.

(i.) In our first example we specify a constitutive map by $\mathfrak{H}(dJ) = J$ for all $J \in E(M, \mathbb{R}^n)$. The following calculation is easily verified :

$$\begin{aligned}
 (7.1) \quad & \int_M dJ \cdot dL \, \mu(J) \\
 &= \int_M \langle \Delta(J)J, L \rangle \mu(J) + \int_{\partial M} \langle dJ(N), l \rangle i_N \mu(J) \\
 &= \int_M \text{tr } A(dL, dJ) \mu(J) \\
 &= \int_M \text{tr } \nabla(J)X(L, J) \mu(J) \\
 &= \int_M \text{div}_J X(L, J) \mu(J) \\
 &= \int_{\partial M} \langle N(j), L \rangle \mu(j) \\
 &= D\left(\int_M \mu(J)\right)(L) ,
 \end{aligned}$$

for all $J \in E(M, \mathbb{R}^n)$ and for all $L \in C^\infty(M, \mathbb{R}^n)$.

Introducing the volume function

$$(7.2) \quad \mathfrak{V} : E(M, \mathbb{R}^n) \rightarrow \mathbb{R} ,$$

assigning to any $J \in E(M, \mathbb{R}^n)$ the volume

$$(7.3) \quad \mathfrak{V}(J) = \int_M \mu(J) ,$$

we have

$$(7.4) \quad D\mathfrak{V}(J)(L) = \int_M dJ \cdot dL \, \mu(J) ,$$

with $N(j) := TJN$ together with the following notions $l := L|_{\partial M}$ and $j := J|_{\partial M}$.

The above calculation shows

$$(7.5) \quad \Phi(dJ) = \Delta(J)J = 0 , \quad \forall J \in E(M, \mathbb{R}^n)$$

and

$$(7.6) \quad \varphi(dJ) = N(j) , \quad \forall J \in E(M, \mathbb{R}^n) .$$

Clearly $l = z$ with $z \in \mathbb{R}^n$ implies

$$(7.7) \quad \int_M \langle N(j), z \rangle \mu(j) = 0, \quad \forall z \in \mathbb{R}^n.$$

With the notations of the previous section this shows that in this example

$$(7.8) \quad \varphi = \varphi_{\mathbb{R}^n}.$$

The map h determined by \mathfrak{H} in this case is thus given by

$$(7.9) \quad N(J) = \Delta(j)h(dJ), \quad \forall J \in E(M, \mathbb{R}^n) \text{ and } j := J|_{\partial M}.$$

(ii.) Next let us turn our attention to h_{∂} on $E(\partial M, \mathbb{R}^n)$, given by $h_{\partial}(dj) = j$, for all $j \in E(\partial M, \mathbb{R}^n)$. One easily verifies the following set of equations

$$(7.10) \quad \begin{aligned} \int_{\partial M} \langle \Delta(j)j, l \rangle \mu(j) &= \int_{\partial M} dj \cdot dl \mu(j) \\ &= \int_{\partial M} (\operatorname{div}_j X(l, j) + \theta(l, j) \cdot H(j)) \mu(j) \\ &= D\left(\int_{\partial M} \mu(j)\right)(l). \end{aligned}$$

Defining the area function

$$(7.11) \quad \mathfrak{A} : E(\partial M, \mathbb{R}^n) \rightarrow \mathbb{R},$$

sending any $j \in E(\partial M, \mathbb{R}^n)$ into

$$(7.12) \quad \mathfrak{A}(j) = \int_{\partial M} \mu(j),$$

we have

$$D\mathfrak{A}(j)(l) = \int_{\partial M} dj \cdot dl \mu(j)$$

for all variables of $D\mathfrak{A}$. The constitutive map h_{∂} determines a map \mathfrak{H}_{∂} given by

$$(7.13) \quad 0 = \Delta(J)\mathfrak{H}_{\partial}(dJ), \quad \forall J \in E(M, \mathbb{R}^n),$$

together with

$$(7.14) \quad d\mathfrak{H}_J(dJ)(N) = \Delta(j)j = H(j) \cdot N(j),$$

for each $J \in E(M, \mathbb{R}^n)$ and $j := J|_{\partial M}$. The function $H(j)$ is the mean curvature, that is the trace of $W(j)$.

(iii.) Next let us consider quite another influence of the boundary by looking at the map $\mathfrak{h}_J: O_{\partial}/\mathbb{R}^n \rightarrow C^\infty(\partial M, \mathbb{R}^n)$ given by $\mathfrak{h}_J(dj) = N(j)$, for all $j \in O_{\partial}$. Then the formula

$$(7.15) \quad \begin{aligned} \Delta(j)N(j) &= \delta dN(J) \\ &= \delta dj W(j) \\ &= -dj \operatorname{grad}_j H(j) + (\operatorname{tr} W(j)^2) \cdot N(j) \end{aligned}$$

holds for any $j \in O_{\partial}$. Let us point out that $\Delta(j)N(j) \neq 0$ even if $j(\partial M) \subset \mathbb{R}^n$ is minimal, that is to say even if $H(j) = \text{const.}$

In the special case of $\dim \partial M = 2$ a topological constant, the Euler characteristic $\chi(\partial M)$, enters the constitutive law F determined by $N(j)$ for each $j \in O_{\partial}$. It is hidden in the formula

$$(7.16) \quad \begin{aligned} F(j)(N(j)) &= \int_{\partial M} \langle \Delta(j)N(j), N(j) \rangle \mu(j) \\ &= \int_{\partial M} \operatorname{tr} W(j)^2 \mu(j). \end{aligned}$$

This is seen by using the Cayley Hamilton theorem (cf.[Gr]) and the Gauss Bonnet theorem (cf.[G,H,V]), applied to the right hand side of equation (7.16), which yields

$$(7.17) \quad F(j)(N(j)) = -4\pi \chi(\partial M) + \int_{\partial M} H(j)^2 \mu(j).$$

8. A general decomposition of constitutive laws

In this section we will exhibit a decomposition of the constitutive map \mathfrak{H} based on the examples of the previous section.

We will show that $D\mathfrak{D}$ and $R^* D\mathfrak{A}$ multiplied with appropriate \mathbb{R} -valued maps are all part of any constitutive law F defined on $E(M, \mathbb{R}^n)$.

To get the full decomposition of a given constitutive law we broaden our scope a little and introduce first of all the Hilbert space A_j consisting of all maps $\gamma_1, \gamma_2 : TM \rightarrow \mathbb{R}$ linear on the fibres of TM for which the right hand side of

$$(8.1) \quad G_{\mathbb{R}^n}^{\partial}(dj)(\gamma_1, \gamma_2) := \int_{\partial M} \gamma_1 \cdot \gamma_2 \mu(j)$$

exists. Clearly $dh_N(dJ)$, dj and $dN(j)$ all belong to A_j and are generically linearly independent. The set O_3 of all $J \in E(M, \mathbb{R}^n)$ for which these three differentials are linearly independent form a dense open set. Let $j := J|_{\partial M}$. In special case $j(\partial M)$ is a $(n-1)$ -sphere in \mathbb{R}^n however, $N(j)$ is a real multiple, r say, of j and $h_N(dJ)$ is hence $\frac{r}{(n-1)} \cdot j$. In the case of linear independence the three above mentioned differentials are in general (with respect to $G_{\mathbb{R}^n}(dj)$) not orthogonal to each other, however. We might orthogonalize them by using the method of Schmidt. For each $J \in O_3$ we then split the differential of $h(dJ)$ into components along span of the three mentioned differentials and a component perpendicular to them.

Next we extend all maps $h_N(dJ)$, j and $N(j)$ to all of M in the following way :

Given $f \in C_j^{\infty}(\partial M, \mathbb{R}^n)$ we solve the following Višik problem (cf. [Hö]) :

$$(8.2) \quad \begin{aligned} \Delta(J)f_M &= 0 \\ df_M(N) - \Delta(j)f &= 0, \end{aligned}$$

with $f_M \in C^{\infty}(M, \mathbb{R}^n)$ and $J \in E(M, \mathbb{R}^n)$ and where $j := J|_{\partial M}$. All the splittings and extensions done to construct j_M and $N(j)_M$ depend smoothly on $j \in E(\partial M, \mathbb{R}^n)$. Let moreover h_N be given by

$$(h_N(dJ))_M = J, \quad \forall J \in E(M, \mathbb{R}^n).$$

The above mentioned decomposition of \mathfrak{H} is then described in the following theorem :

Theorem 8.1 :

Let F be a constitutive law on $E(M, \mathbb{R}^n)$ determined by $\mathfrak{H} \in C^\infty(E(M, \mathbb{R}^n)/\mathbb{R}^n, C^\infty(M, \mathbb{R}^n))$. Then \mathfrak{H} determines uniquely three smooth maps

$$(8.3) \quad a_1, a_2, a_3 : O_3 \subset E(M, \mathbb{R}^n)/\mathbb{R}^n \rightarrow \mathbb{R}$$

and two smooth maps

$$\mathfrak{h}, \mathfrak{h}^2 : O_3 \subset E(M, \mathbb{R}^n)/\mathbb{R}^n \rightarrow C^\infty(\partial M, \mathbb{R}^n),$$

such that the following splitting holds for any $dJ \in O_3 \subset E(M, \mathbb{R}^n)$

$$(8.4) \quad \mathfrak{h}(dJ) = a_1(dJ) \cdot \mathfrak{h}_N(dJ) + a_2(dJ)j + a_3(dJ) \cdot N(j) + \mathfrak{h}_2(dJ)$$

with $j := j/\partial M$. The differential $d\mathfrak{h}_2(dJ)$ is orthogonal with respect to $G(dj)_{\mathbb{R}^n}^\partial$ to the span of $d\mathfrak{h}_N(dJ)$, dj and $dN(j)$.

The map $\mathfrak{H}(dJ)$ decomposes for each $J \in O_3$ accordingly into

$$(8.5) \quad \mathfrak{H}(dJ) = a_1(dJ) \cdot J + a_2(dJ) \cdot J_M + a_3(dJ) \cdot N_M(dJ) + \mathfrak{H}_2(dJ),$$

with $j := J|\partial M$ and where $\mathfrak{H}_2(dJ)$ is such that (8.10) holds.

△

9. The rotational symmetry

Finally we investigate the effect of the $SO(n)$, the symmetry group of the oriented Euclidean vector space $(\mathbb{R}^n, <, >, \underline{\Delta})$ to a constitutive law. In particular let us characterize those which are invariant under $SO(n)$.

As mentioned in the first section we have the operation

$$\mathfrak{s} : SO(n) \times E(M, \mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

sending any $g \in SO(n)$ and any $J \in E(M, \mathbb{R}^n)$ into

$$\mathfrak{s}(g, J) = g \circ J.$$

Let $\mathfrak{s}_g(J) := g \circ J$ for each pair $g \in SO(n)$ and each $J \in E(M, \mathbb{R}^n)$.

The induced operation \mathfrak{s}_T on $TE(M, \mathbb{R}^n)$ is therefore determined by

$$\mathfrak{s}_T : SO(n) \times E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \rightarrow E(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n)$$

$$\mathfrak{s}_T(g, (J, L)) = (g \circ J, g \circ L),$$

for all $g \in SO(n)$, for all $J \in E(M, \mathbb{R}^n)$ and for all $L \in C^\infty(M, \mathbb{R}^n)$.

Invariance under $SO(n)$ of a given constitutive law F with \mathfrak{H} means that the following equation

$$(9.1) \quad F(g \circ J)(g \circ L) = F(J)(L)$$

holds for all the variables of F and for any $g \in SO(n)$. Thus the constitutive map satisfies

$$(9.2) \quad \int_M d\mathfrak{H}(g \circ dJ) \cdot d(g \circ L) \mu(g \circ J) = \int_M d\mathfrak{H}(dJ) \cdot dL \mu(J)$$

or reformulated

$$(9.3) \quad \int_M \text{tr} [(\bar{g}^{-1} A(d\mathfrak{H}(g \circ dJ), g \circ dJ) - A(d\mathfrak{H}(dJ), dJ)) \cdot \tilde{A}(dL, dJ)] \mu(J) = 0$$

with $\bar{g} = (dJ)^{-1} \circ g$. Thus $\bar{g} \in SO(T_p M)$, the special orthogonal group of $T_p M$ for each $p \in M$. The volume μ is invariant under the action \mathfrak{s}_T . Hence

$$(9.4) \quad \int_M d(\bar{g}^{-1} \circ \mathfrak{H}(g \circ dJ) - \mathfrak{H}(dJ)) \cdot dL \mu(J) = 0,$$

holds for all $g \in SO(n)$, for all $J \in E(M, \mathbb{R}^n)$ and for all $L \in C^\infty(M, \mathbb{R}^n)$. From the last equation and from the general procedure of representing differentials via embeddings we read off

Proposition 9.1 :

Given a constitutive law F with constitutive map $\mathfrak{H} \in C^\infty(E(M, \mathbb{R}^n), C^\infty(M, \mathbb{R}^n))$ then F is invariant under $SO(n)$ iff

$$(9.5) \quad g^{-1} \circ d\mathfrak{H}(g \circ dJ) = d\mathfrak{H}(dJ), \quad \forall G \in SO(n).$$

The validity of this equation is equivalent with the following identity

$$(9.6) \quad A(d\mathfrak{H}(g \circ dJ), g \circ dJ)(p) = A(d\mathfrak{H}(dJ), dJ)(p)$$

holding for all $g \in SO(n)$, for all $p \in M$ and for all $J \in E(M, \mathbb{R}^n)$. Moreover the stress tensor $T(J)$ determined by \mathfrak{H} is invariant under $SO(n)$ for any $J \in E(M, \mathbb{R}^n)$.

△

Let F be a $SO(n)$ -invariant constitutive law. (9.5) implies that for each $c \in \mathfrak{so}(n)$

$$(9.7) \quad \text{cod} \mathfrak{H}(dJ) = dD\mathfrak{H}(dJ)(\text{cod} J),$$

with D the derivative on $E(M, \mathbb{R}^n)/\mathbb{R}^n$. Here $\mathfrak{so}(n)$ is the Lie-algebra of $SO(n)$.

Based on (9.7) it is a matter of routine to show that in case \mathfrak{H} is invariant under $SO(n)$ then $c \circ d\mathfrak{H}(dJ) = 0$ for any $c \in \mathfrak{so}(n)$ and any $J \in E(M, \mathbb{R}^n)$. In case of $\dim M = 3$ this means that the $SO(n)$ -invariance of $\mathfrak{H}(J)$ yields $\mathfrak{H}(dJ) = \text{const.}$ and in turn that $F = 0$.

Splitting $A(d\mathfrak{H}(dJ), dJ)$ with respect to $m(J)$ into symmetric and skew-symmetric part and doing the same for $A(dD\mathfrak{H}(dJ)(\text{cod} J), dJ)$ and using (9.7) yields

$$(9.8) \quad \bar{c} \circ B(d\mathfrak{H}(dJ), dJ)(p) + \bar{c} \circ C(d\mathfrak{H}(dJ), dJ)(p) \\ = B(dD\mathfrak{H}(dJ)(\text{cod} J), dJ)(p) + C(dD\mathfrak{H}(dJ)(\text{cod} J), dJ)(p)$$

with $\bar{c} \in \mathfrak{so}(T_p M)$ for each fixed $p \in M$. Here $\mathfrak{so}(T_p M)$ denotes the Lie Algebra of $SO(T_p M)$. If $\text{tr } A(dD\mathfrak{H}(dJ)(\text{cod} J), dJ) = 0$ then (9.8) yields

$$(9.9) \quad \text{tr } \bar{c} \circ C(d\mathfrak{H}(dJ), dJ)(p) = 0, \quad \forall c \in \mathfrak{so}(n), \forall p \in M \\ \text{and } \forall J \in E(M, \mathbb{R}^n).$$

and vice versa. This implies the following theorem :

Theorem 9.2 :

For any $SO(n)$ -invariant constitutive law with (smooth) constitutive map \mathfrak{H} the following are equivalent:

$$(9.10) \quad C(d\mathfrak{H}(dJ), dJ) = 0, \quad \forall J \in E(M, \mathbb{R}^n)$$

$$(9.11) \quad \text{tr } A(dD\mathfrak{H}(dJ)(c \odot J), dJ) = 0 \quad \forall J \in E(M, \mathbb{R}^n)$$

(9.12) The stress tensor $T(J)$ associated with \mathfrak{H} is symmetric for each configuration J .

Δ

Let us illustrate (9.11) somewhat. In doing so we proceed similar as in section 8 and take the components of $d\mathfrak{H}(g \cdot dJ)$ and $d\mathfrak{H}(dJ)$ along dJ , this is to say we have the splitting

$$(9.13) \quad d\mathfrak{H}(g \cdot dJ) = \hat{\Pi}(g \cdot dJ) \cdot dJ + d\hat{\mathfrak{H}}_1(g \cdot dJ)$$

and

$$(9.14) \quad d\mathfrak{H}(dJ) = \Pi(dJ) \cdot dJ + d\mathfrak{H}_1(dJ),$$

where both $d\hat{\mathfrak{H}}_1(dJ)$ and $d\mathfrak{H}_1(dJ)$ are orthogonal to dJ with respect to $G_{\mathbb{R}^n}(dJ)$.

Both Π and $\hat{\Pi}$ can be regarded as internal pressures. All the maps in (9.13) and (9.14) are smooth. We therefore find

$$(9.15) \quad \begin{aligned} \int_M \text{tr } A(d\mathfrak{H}(g \cdot dJ), dJ) \mu(J) &= n \cdot \hat{\Pi}(g \cdot dJ) \cdot \mathfrak{V}(J) \\ &= n \cdot \Pi(dJ) \cdot \mathfrak{V}(J) \cdot \text{tr } g + \int_M \text{tr } A(g \cdot d\mathfrak{H}_1(dJ), dJ) \mu(J) \end{aligned}$$

and in turn

$$(9.16) \quad \int_M \text{tr } A(dD\mathfrak{H}(dJ)(c \cdot dJ), dJ) = n \cdot \mathfrak{V}(J) \cdot D\hat{\Pi}(dJ)(c \cdot dJ).$$

Hence (9.11) requires

$$(9.17) \quad D\hat{\Pi}(dJ)(c \cdot dJ) = 0, \quad \forall J \in E(M, \mathbb{R}^n) \text{ and } \forall c \in \mathfrak{so}(n),$$

showing that $\hat{\Pi}$ is up to the first order invariant under $SO(n)$.

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